MATH 122B: HOMEWORK 4

Suggested due date: August 29nd 2016

(1) Compute $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)}, \quad \int_0^\infty \frac{\sqrt{x}dx}{(x+1)^2}$

(2) Compute $\displaystyle\int_0^\infty \frac{\operatorname{Log} x dx}{x^4+1}$

(3) Compute $\displaystyle\int_{-\infty}^\infty \frac{e^{ax}dx}{\cosh(x)}$, $|a|<1$.

(4) Show that $\displaystyle\int_0^\infty \frac{x^{p-1}}{1+x}dx = \frac{\pi}{\sin(p\pi)}$, $0<p<1$.

(5) Show that $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{a}\coth \pi a$, a is real and noninteger.

(6) Find the sum $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$.

(7) Consider the transformation $w=f(z)$ where $f(z)$ is analytic at z_0 and $f'(z_0)\neq 0$. Prove that under this transformation, the tangent at z_0 to any curve C in the z plane passing through z_0 is rotated through the angle $\alpha = \arg f'(z_0)$.

(8) Find a bilinear transformation that maps points $z=0,-i,-1$ into $w=i,1,0$ respectively.

Solutions

(1) Using a key hole contour, one can derive

$$\int_0^\infty x^a f(x)dx = \frac{2\pi i}{1-e^{2\pi i a}} \sum R(z^a f(z), z_j); \quad -1<1<1.$$

So

$$\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\sqrt{2}\pi}{2}$$

and

$$\int_0^\infty \frac{\sqrt{x}dx}{(x+1)^2} = \frac{\pi}{2}.$$

(2) There is a singularity at $z=0$, so we use an indented semicircle contour. It can be shown that the circular portions both go to zero. Now for the line coming from $-R$ to r, we parametrize as $z = xe^{i\pi}$, with x going from R to r so

$$\begin{aligned}\int_C \frac{\operatorname{Log} z dz}{z^4+1} &= \int_r^R \frac{\operatorname{Log} x dx}{x^4+1} + \int_R^r \frac{\operatorname{Log}(x)+i\pi}{x^4+1} e^{i\pi} dx \\ &= 2\int_r^R \frac{\operatorname{Log} x dx}{x^4+1} + i\pi \int_r^R \frac{dx}{x^4+1}\end{aligned}$$

Computing the residue, we have

$$\begin{aligned}\int_C \frac{\operatorname{Log} z dz}{z^4+1} &= 2\pi i(\frac{\operatorname{Log}(e^{i\pi/4})}{4e^{i3\pi/4}} + \frac{\operatorname{Log}(e^{3i\pi/4})}{4e^{i9\pi/4}}) \\ &= -\frac{\pi^2}{8}(e^{-i3\pi/4} + 3e^{-i9\pi/4}) \\ &= -\frac{\pi^2}{8}e^{-i3\pi/2}(e^{i3\pi/4} + 3e^{-i3\pi/4}) \\ &= -\frac{\pi^2}{8}i(2\cos(3\pi/4) + 2e^{-i3\pi/4}) \\ &= -\frac{i\pi^2}{8}(-\sqrt{2} - \sqrt{2} - \sqrt{2}i)\end{aligned}$$

Taking the real part, we get $-\frac{\pi^2\sqrt{2}}{16}$.

(3) Changing variables $t = e^x$, $\frac{dt}{t} = dx$, so

$$\int_{-\infty}^{\infty} \frac{e^{ax}dx}{\cosh(x)} = 2\int_0^{\infty} \frac{t^a}{t^2+1}dt$$

so using the same method as 1, we have

$$\begin{aligned}2\int_0^{\infty} \frac{t^a}{t^2+1}dt &= \frac{4\pi i}{1-e^{2\pi ia}}(\frac{e^{ia\pi/2}}{2i} - \frac{e^{ia3\pi/2}}{2i}) \\ &= 2\pi\frac{\sin(a\pi/2)}{\sin(\pi a)}\end{aligned}$$

(4) Using the same method,

$$\begin{aligned}\int_0^{\infty} \frac{x^{p-1}}{1+x}dx &= \frac{2\pi i}{1-e^{2\pi ip}}((e^{i(p-1)\pi}) \\ &= -\frac{2\pi i}{e^{-ip\pi} - e^{ip\pi i}}) \\ &= \frac{\pi}{\sin(p\pi)}\end{aligned}$$

(5) Let $f(z) = \frac{1}{z^2+a^2}$. Then

$$\begin{aligned}\sum_{n\in\mathbb{Z}} f(n) &= -(\operatorname{Res}(\frac{\pi\cos(\pi z)}{\sin(\pi z)}f(z), ai) + \operatorname{Res}(\frac{\pi\cos(\pi z)}{\sin(\pi z)}f(z), -ai)) \\ &= -(\frac{\pi\cos(a\pi i)}{2ai\sin(a\pi i)} + \frac{\pi\cos(a\pi i)}{2ai\sin(a\pi i)} \\ &= \frac{\pi}{a}\frac{\cosh(a\pi)}{\sinh(a\pi)}\end{aligned}$$

(6) Let $f(z) = \frac{1}{x^2}$. Then

$$\begin{aligned}2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} &= -\operatorname{Res}(\frac{\pi}{z^2\sin(\pi z)}, 0) \\ &= \frac{\pi^2}{6}\end{aligned}$$

(7) Let $z(t)$ be a curve. Then $w(t) = f(z(t))$ so the tangent vector is given by $w'(t) = f'(z)z'(t)$. Since $\text{Arg}(zw) = \text{Arg}(z) + \text{Arg}(w)$, we have

$$\text{Arg}(w'(t)) = \text{Arg}(f') + \text{Arg}(z'(t))$$

so the tangent vector rotates by f', for any curve, hence is conformal.

(8) Using cross ratios or solving a linear system of equations, we get $w = \dfrac{iz + i}{1 - z}$.